2017 International Conference on Computational Science and Computational Intelligence

Calculating Persistence Homology of Attractors in High-Dimensional Chaotic Systems

Anda Xiong and Amir Assadi
Department of Math
University of Wisconsin Madison
Madison, U.S.A
axiong23@wisc.edu; amir.h.assadi@gmail.com

Yueqi Cao
Department of Math
Beijing Institute of Technology
Beijing, China

***Abstract*—While persistent homology is a basic but useful tool for describing the topological features of spaces, it has rarely been applied to describe the chaotic attractor whose shape could be very intriguing and complex. Especially for high-dimensional chaotic attractors, on one hand it reveals the evolution and stability of dynamical systems yet on the other hand, since it is a natural restriction for human researchers that they are unable to see more than three dimensionalities, methods for categorizing high dimensional attractors are lacking. Therefore, in this article we propose the method of applying persistent homology calculation to high-dimensional chaotic attractors and give several representative examples from different genres of chaotic systems for whose topological structure being displayed.**

***Keywords*-Nonlinear Dynamics; Chaos; Persistent Homology; Strange Attractors; Topology**

I. Introduction

Studying nonlinear dynamical systems is complex, and such complexity increases with dimensionality. Currently most commonly studied chaotic systems are in low dimension, such as two-dimensional mappings like Henon mapping and Lozi mapping, and three-dimensional ODE systems like the famous Lorenz system and Chua's circuit. On contrary to researchers' intensity on two or three dimensional chaotic systems, only the hyper-Rossler system, a four-dimensional ODE systems is a better known hyperchaotic system, where hyperchaos means that the Lyapunov exponent is positive for at least two dimensions. The Lyapunov exponent characterizes the divergence of trajectories of two points initially close to each other, and a positive Lyapunov exponent indicates that two initial points in the phase space separates from each other exponentially, which is a criterion for chaos.

Dissipation in physical systems appears as attractions in phase space, and attractions could be classified into three types: equilibrium point, limit cycle/ torus, and strange attractors. While the first two types of attractions are relatively plain in the sense of geometry, strange attractors fascinate researchers by their complex shapes. More importantly, strange attractor reveals the evolution of dynamical systems, where the attractor is a stable state that systems evolve toward. While researchers are familiar with attractors in two or three dimensions since humans are restricted from visualizing beyond three dimensions, in real physical world, however, dynamical systems are usually affected by many factors therefore their phase space should have a high dimensionality for each variable corresponding a physical quantities in the system, such as temperature, viscosity, and many else. Therefore, if researchers want to step forward from simplified ODE equations to more realistic cases, it is necessary to develop a method that enables researchers to categorize high dimensional attractors. Considering such need, in this article we calculate the persistence homology of several kinds of chaotic attractors in order to manifest our proposition of this method.

II. Based Theory of Method on Persistent Homology

A. Simplicial Complexes and Simplicial Homology

For a detailed description of the simplicial homology theory, we recommend the classical textbook of (Munkres, 1984). Here we introduce the basic definitions of simplicial homology that are sufficient for our research.

Let $a_0, a_1, \cdots, a_d$ be $(d+1)$-points in the d-dimensional Euclidean space $\mathbb{R}^d$. The $(d+1)$-points are said to be in general position if the vectors $a_1 - a_0, a_2 - a_0, \cdots, a_d - a_0$ are linearly independent. A d-simplex is the convex hull of the $(d+1)$-points. Equivalently we can express the d-simplex spanned by $a_0, a_1, \cdots, a_d$ as the set

$$\{x = \lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_d a_d \in \mathbb{R}^d | \lambda_0 + \lambda_1 + \cdots + \lambda_d = 1\}$$

By definition a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

A simplicial complex K is a finite collection of simplices that satisfy the following two rules

1) If σ is a simplex in K, then all the faces of σ are in K, where the faces of σ are the simplices spanned by vertices $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ of σ;
2) If σ and τ are two simplices in K, then the intersection of σ and τ is either empty or their common face.

978-1-5386-2652-8/17 $31.00 © 2017 IEEE
DOI 10.1109/CSCI.2017.24

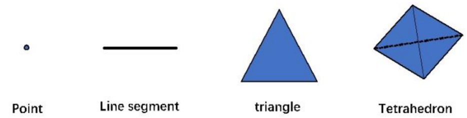

Figure 1. Simplices in varies dimensions

The dimension of K is the maximum dimension of its simplices. A subcomplex of K is a subset $L \subset K$ that is also a simplicial complex. For example, let $K = \{[v_0], [v_1], [v_2], [v_0, v_1], [v_0, v_2], [v_1, v_2], [v_0, v_1, v_2]\}$ be a 2-dimensional simplicial complex. $L = \{[v_0], [v_1], [v_0, v_1]\}$ is a subcomplex of K.

Let K be a simplicial complex. Let $C_i(K)$ be the vector space spanned by the i-simplices of K over coefficient field $\mathbb{F}_2$. Here $\mathbb{F}_2$ is the field consisting of only two elements. Define the boundary map $\partial_i : C_i(K) \to C_{i-1}(K)$, where ∂_i acts on the basis σ as

$$\partial_i(\sigma) = \sum_{\tau \, is \, an \, (i-1)-face \, of \, \sigma} \tau$$

and ∂_i extends linearly on whole $C_i(K)$. We thus have the following sequence

$$0 \longrightarrow \cdots \xrightarrow{\partial_{i+2}} C_{i+1}(K) \xrightarrow{\partial_{i+1}} C_i(K) \xrightarrow{\partial_i} C_{i-1}(K) \xrightarrow{\partial_{i-1}} \cdots \longrightarrow 0$$

The boundary maps satisfy the relation $\partial_i \circ \partial_{i+1} = 0$. Therefore, the image of ∂_{i+1} is a subset of the kernel of ∂_i. Define the i-th homology group $H_i(K)$ to be the quotient space

$$H_i(K) := \mathit{kernel}(\partial_i)/\mathit{image}(\partial_{i+1})$$

The dimension of $H_i(K)$ is called the i-th Betti number. An important property of Betti numbers is that they are topological invariants, i.e. they are invariant under topology transformation. Hence Betti numbers are useful tools to distinguish spaces.

Geometrically, one can portrait n-th Betti number as an n-dimensional hole in the space. Therefore, by calculating the Betti number in different dimensions of any objects one obtains the topological structure on it. It is a very useful tool to portrait the geometry computationally by a software that will be introduced on next chapter.

B. Using the Software

Javaplex is a powerful computational topology package to compute persistent homology (Adams, 2011). All the present results are computed using Javaplex. We input data points of attractors that we want to study into the program and run the program. The program can detect the Betti number in different dimensions therefore it gives us the geometrical information on the topology of attractors. It is worth noting that the result is valid for macroscopic geometry only. For some attractors they have holes in very small scale that is easy to be neglected, and such error is tolerated because we are only concerning the general topology of attractors, since it gives the most of information one needs.

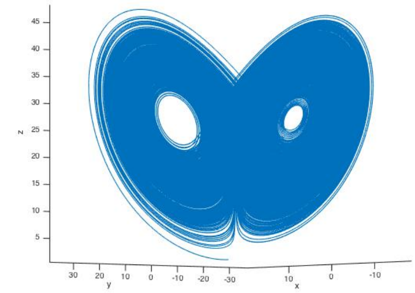

Figure 2. The Lorenz attractor

III. Calculating the Persistent Homology of Attractors

We hereby examine the topological structure of several most representative chaotic attractors. The first one is the most well known chaotic system, the Lorenz system (Lorenz, 1963).

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

Normally the three parameters are chosen to be $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. Under this condition, almost all points in the space of initial conditions will move towards and finally remain on the chaotic Lorenz attractor as shown in figure 2. We use the program to compute the persistent homology of such attractor. As indicated by the figure 3, we observe that there is Betti number of 1 in dimension 0 which means that the attractor is single path connected. Betti number of 2 in dimension 1 means that there are two of one dimensional holes, which are two loops on the Lorenz attractor, and this do follows the Lorenz attractor displayed in 3 dimension as shown in figure 2.

While we have studied the topology of many attractors by such method, and obtains interesting results, we give

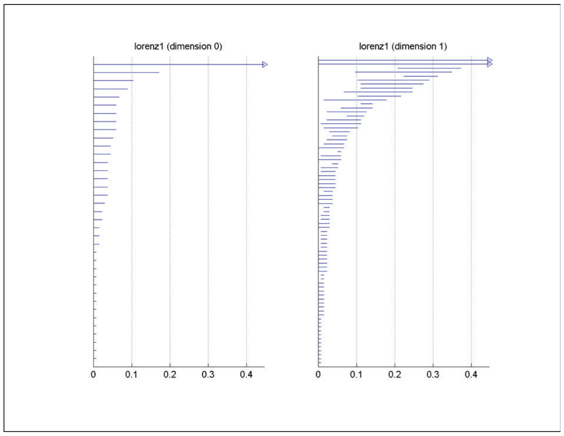

Figure 3. The persistent homology of the Lorenz attractor

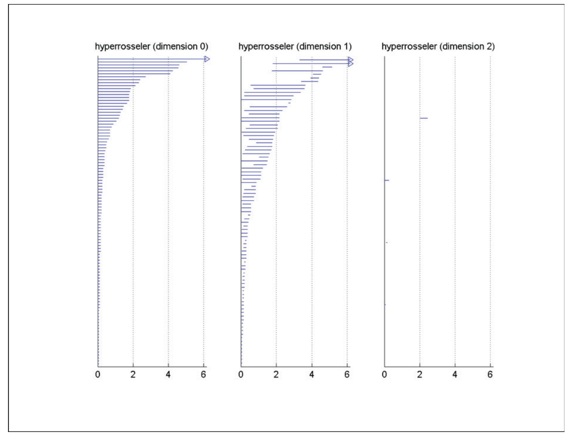

Figure 4. The persistent homology of the Hyper-Rosseler attractor

one more example of the Hyper-Rossler chaotic attractor (Rossler, 1979) which is the most well known attractor beyond three-dimensions.

$$\frac{dx}{dt} = -y - x$$

$$\frac{dy}{dt} = x + ay + w$$

$$\frac{dz}{dt} = b + xz$$

$$\frac{dw}{dt} = -cz + dw$$

with parameters to be a=0.25, b=3, c=0.5 and d=0.05. This is a four dimensional system and therefore cannot be visualised. Therefore we use the program to detect its topology, which is shown in figure 4.While it is a four dimensional attractors, it has only two one-dimensional homology, and its two-dimensional homology vanishes in macroscopic scale.

IV. Conclusion

We propose the method of persistent homology to study the geometric structure of chaotic attractors that represent the final state of evolution of dynamical systems, especially in high dimensionalities for where the persistent homology gives observation that cannot be made by direct visualizations. While though seldom there has been previous works using the same method, such as (Mittai, 2017) and (Maletic, 2016), they are not aware of the importance of applying such method to detect high dimensional attractors which cannot be visualised, but rather discussing the general features of homology. We believe such method could be widely applied to different areas of complex system in high dimensions.

References

[1] Munkres J. R. [1984] “Elements of algebraic topology”*Menlo Park: Addison-Wesley.*

[2] Adams H, Tausz A. [2011] Javaplex tutorial” *Dostopno na: https://code. google. com/p/javaplex/wiki/Tutorial*

[3] Lorenz, E. N. [1963] Deterministic nonperiodic flows,” *J. Atmos. Sci.* **20**(2), 130–141.

[4] O. E. Rssler. [1979] An equation for hyperchaos, Physics Letters A, 71, 155-157.

[5] Mittal K, Gupta S. [2017] Topological characterization and early detection of bifurcations and chaos in complex systems using persistent homology.[J]. Chaos An Interdisciplinary Journal of Nonlinear Science.

[6] Maletic S, Zhao Y, Rajkovic M. [2016] Persistent topological features of dynamical systems[J]. Chaos An Interdisciplinary Journal of Nonlinear Science.